REPRESENTATION OF SOLUTIONS OF DYNAMICS PROBLEMS OF HIGHLY INTENSIVE HEAT TRANSFER

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Solutions of a hyperbolic heat conduction equation, which are represented by a class of special functions, are obtained for generalized boundary conditions of the first-third kind by the Laplace transform method.

The practical realization of highly intensive heat transfer in pulse [1] and laser [2] engineering units, in laser metal treatment [3], in plasma coating processes [4], and also in energy channels [5] evoked interest in studying this phenomenon.

The theoretical foundation for the relaxation model of heat conduction given by A. V. Lykov [6] on the basis of the general theory of irreversible thermodynamics resulted in numerous investigations of the highly intensive heat transfer phenomenon in bodies which is described, for the one-dimensional case, by the equations

$$q+\tau_{rq}\frac{\partial q}{\partial\tau}=-\lambda\frac{\partial\Theta}{\partial y}, \tag{1}$$

$$\frac{\partial\Theta}{\partial\tau}+\tau_{rq}\frac{\partial^2\Theta}{\partial\tau^2}=a\frac{\partial^2\Theta}{\partial y^2}. \tag{2}$$

Obtaining the solution of (1) and (2) by analytical methods for different boundary conditions and developing engineering computational formulas were restricted by three factors:

1. The problem is reduced to a class of transforms of the type

$$F(s)=\frac{\exp\left[-y\sqrt{(s+\alpha)(s+\beta)}\right]}{s^i\left(\sqrt{(s+\alpha)(s+\beta)}\right)^n} \tag{3}$$

when applying an integral transform method, represented in the handbook literature on the Laplace transform in limited quantity.

2. Available solutions of individual boundary-value problems contain single and multiple integrals of a special class:

$$\int\limits_0^y\tau^p\exp(-\rho\tau)\frac{I_n(\sigma\sqrt{\tau^2-v^2})}{(\sqrt{\tau^2-v^2})^i}\,dv,\quad\int\limits_y^\tau t^p\exp(-\rho t)\frac{I_n(\sigma\sqrt{t^2-y^2})}{(\sqrt{t^2-y^2})^i}\,dt, \tag{4}$$

which are difficult to use in engineering computations.

3. In formulating the boundary conditions for (1) and (2) the majority of authors ignore the relaxation nature of the heat transfer; boundary conditions of the first-third kind were described by classical relationships, which does not permit the use of the results obtained.

It is pertinent to note that (2), referred to the class of hyperbolic equations of mathematical physics, not only describes thermal phenomena but also the propagation of waves originating on a different physical basis: thermoelastic [7,8], electric [9], and shocks in fluids [10]. Moreover, diffusion phenomena in gases [11] and moisture transfer in capillary porous bodies [6] are described by the hyperbolic equation (2). Problems of evaluating the integrals (4) and finding the originals from the transforms (3) also occur in solving the problems mentioned. It hence becomes clear that the solution of this problem is of great value for the analysis of a large class of heat, moisture, mechanical and electrical energy, etc. transport phenomena.

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TABLE 1. Representation of Integrals by V-functions

Integral	Representation
$\int_y^\tau e^{-\rho t} I_0(\sigma\sqrt{t^2-y^2})\,dt$	$\frac{1}{\sqrt{\rho^2-\sigma^2}} e^{(\sigma-\rho)\tau}(V_{1,1}-V_{1,2})$ (5)
$\int_y^\tau te^{-\rho t} I_0(\sigma\sqrt{t^2-y^2})\,dt$	$\frac{1}{\rho^2-\sigma^2} e^{(\sigma-\rho)\tau}\left\{\frac{\rho(y\sqrt{\rho^2-\sigma^2}+1)}{\sqrt{\rho^2-\sigma^2}}V_{1,1}+\right.$ $+\frac{\rho(y\sqrt{\rho^2-\sigma^2}-1)}{\sqrt{\rho^2-\sigma^2}}V_{1,2}-$ $\left.-2\xi\left(\frac{\rho}{\sigma}V_{1,0}+V_{2,0}\right)\right\}$ (6)
$\int_y^\tau e^{-\rho t}\frac{I_1(\sigma\sqrt{t^2-y^2})}{\sqrt{t^2-y^2}}\,dt$	$\frac{1}{\sigma y}\left\{e^{(\sigma-\rho)\tau}(V_{1,1}+V_{1,2}-V_{1,0})-e^{-\rho y}\right\}$ (7)
$\int_y^\tau te^{-\rho t}\frac{I_1(\sigma\sqrt{t^2-y^2})}{\sqrt{t^2-y^2}}\,dt$	$\frac{1}{\sigma}\left\{e^{(\sigma-\rho)\tau}\left[V_{1,0}+\frac{\rho}{\sqrt{\rho^2-\sigma^2}}\times\right.\right.$ $\left.\left.\times(V_{1,1}-V_{1,2})\right]-e^{-\rho y}\right\}$ (8)
$\int_y^\tau e^{-t} I_0(\sqrt{t^2-y^2})\,dt$	$2V_2$ (9)
$\int_y^\tau e^{-t}\frac{I_1(\sqrt{t^2-y^2})}{\sqrt{t^2-y^2}}\,dt$	$\frac{1}{y}(2V_1-V_{1,0}-e^{-y})$ (10)
$\int_y^\tau te^{-t}\frac{I_1(\sqrt{t^2-y^2})}{\sqrt{t^2-y^2}}\,dt$	$2V_2+V_{1,0}-e^{-y}$ (11)
$\int_y^\tau (t-y)e^{-t}\frac{I_1(\sqrt{t^2-y^2})}{\sqrt{t^2-y^2}}\,dt$	$2(V_2-V_1)$ (12)
$\int_0^y e^{-\tau}\left[I_0(\sqrt{\tau^2-v^2})+\right.$ $\left.+\tau\frac{I_1(\sqrt{\tau^2-v^2})}{\sqrt{\tau^2-v^2}}\right]dv$	$1+V_{1,0}-2V_1$ (13)

In the present paper it is proposed to use the mathematical apparatus of special functions [12] to represent integrals of the type (4), to find the transform-original correspondence of (3), and on this basis to construct the solution of relaxation heat conduction boundary-value problems.

The representation of certain integrals and originals of transforms by V-functions is given in Tables 1 and 2. The definition of the functions, their properties, and numerical tables are given in [12]. By definition $V_{i,k} = V_{i,k}(\xi, \eta, c_k)$; for $c_k = 1$ $V_{i,k} \equiv V_i$ while $V_{i,k} \equiv V_{i,0}$ for $c_k = 0$. The following arguments are taken in Tables 1 and 2:

For formulas (5)-(8), (18)-(21)

$$\xi = \frac{\sigma}{2}(\tau + y), \quad \eta = \frac{\sigma}{2}(\tau - y), \quad c_1 = \frac{\sigma}{\rho - \sqrt{\rho^2-\sigma^2}}, \quad c_2 = \frac{1}{c_1};$$

TABLE 2. Laplace Correspondence on the Basis of V-Functions

Laplace transform	Original	
$\frac{1}{s}\exp[-y\sqrt{(s+\alpha)(s+\beta)}]$	$e^{-\beta\tau}(V_{1,1}+V_{1,2}-V_{1,0})$	(14)
$\frac{\exp[-y\sqrt{(s+\alpha)(s+\beta)}]}{\sqrt{(s+\alpha)(s+\beta)}}$	$e^{-\beta\tau}V_{1,0}$	(15)
$\frac{\exp[-y\sqrt{(s+\alpha)(s+\beta)}]}{s\sqrt{(s+\alpha)(s+\beta)}}$	$\frac{1}{\sqrt{\alpha\beta}}e^{-\beta\tau}(V_{1,1}-V_{1,2})$	(16)
$\frac{1}{s}\sqrt{\frac{s+\beta}{s+\alpha}}\times$ $\times\exp[-y\sqrt{(s+\alpha)(s+\beta)}]$	$e^{-\beta\tau}\left[\sqrt{\frac{\beta}{\alpha}}(V_{1,1}-V_{1,2})+V_{1,0}\right]$	(17)
$\frac{1}{s}\exp[-y\sqrt{s(s+2\sigma)}]$	$2V_1-V_{1,0}$	(18)
$\frac{\exp[-y\sqrt{s(s+2\sigma)}]}{s\sqrt{s(s+2\sigma)}}$	$\frac{2}{\sigma}V_2$	(19)
$\frac{1}{s}\sqrt{\frac{s+2\sigma}{s}}\exp[-y\sqrt{s(s+2\sigma)}]$	$4V_2+V_{1,0}$	(20)
$\frac{\sigma^i\exp[-y\sqrt{s(s+2\sigma)}]}{\sqrt{s(s+2\sigma)}\,[\sqrt{s(s+2\sigma)}+s+\sigma]^i}$	$V_{i+1,0},\quad i=0,\ 1,\ 2,\ \ldots$	(21)
$\frac{\exp[-y\sqrt{s(s+1)}]}{s+1}$	$2V_{-1}-V_{1,0}$	(22)
$\frac{s+c}{\sqrt{s(s+1)}}\exp[-y\sqrt{s(s+1)}]$	$\left(c-\frac{1}{2}\right)V_{1,0}+\frac{\xi+\eta}{4\eta}V_{2,0}$	(23)
$\frac{s^2+bs+c}{s\sqrt{s(s+1)}}\exp[-y\sqrt{s(s+1)}]$	$4cV_2+\left(b-\frac{1}{2}\right)V_{1,0}+\frac{\xi+\eta}{4\eta}V_{2,0}$	(24)
$\frac{1}{s^2}\exp[-y\sqrt{s(s+1)}]$	$8V_3+4V_2$	(25)
$\frac{1}{s+\sqrt{s(s+1)}}\exp[-y\sqrt{s(s+1)}]$	$\frac{1}{2}(V_{1,0}+V_{2,0})$	(26)
$\frac{\sqrt{s(s+1)}}{s[s+\sqrt{s(s+1)}]}\exp[-y\sqrt{s(s+1)}]$	$2V_1-\frac{3}{2}V_{1,0}-\frac{1}{2}V_{2,0}$	(27)
$\frac{1}{s+a\sqrt{s(s+1)}}\exp[-y\sqrt{s(s+1)}]$	$\frac{2}{1-a^2}V_{1,1}-\frac{1}{1-a}V_{1,0}$	(28)
$\frac{a\sqrt{s(s+1)}}{s[s+a\sqrt{s(s+1)}]}\times$ $\times\exp[-y\sqrt{s(s+1)}]$	$2V_1+\frac{a}{1-a}V_{1,0}-\frac{2}{1-a^2}V_{1,1}$	(29)
$\frac{1}{a+\sqrt{s(s+1)}}\exp[-y\sqrt{s(s+1)}]$	$V_{1,0}+\frac{2a}{\sqrt{1+4a^2}}(V_{1,2}-V_{1,3})$	(30)
$\frac{1}{s+a}\exp[-y\sqrt{s(s+1)}]$	$V_{1,4}+V_{1,5}-V_{1,0}$	(31)

for (9)-(13)

$$\xi=\frac{\tau+y}{2},\quad \eta=\frac{\tau-y}{2};$$

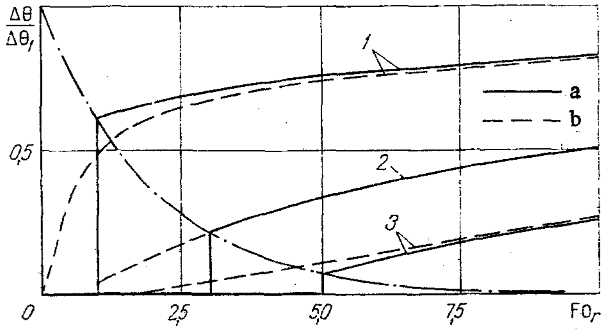

Fig. 1. Change in body temperature for a temperature jump on its surface (relaxation model (a); Fourier model (b)): 1) Y = 1, 2) 3; 3) 5; dash—dot curve $e^{-0.5Fo_r}$.

for (14)-(17)

$$\xi = \frac{\alpha-\beta}{4}(\tau+y), \quad \eta = \frac{\alpha-\beta}{4}(\tau-y),$$

$$c_1 = \frac{\sqrt{\alpha}+\sqrt{\beta}}{\sqrt{\alpha}-\sqrt{\beta}}, \quad c_2 = \frac{1}{c_1};$$

for (22)-(31)

$$\xi = \frac{\tau+y}{4}, \quad \eta = \frac{\tau-y}{4}, \quad c_1 = \frac{1-a}{1+a}, \quad c_2 = -\frac{1}{c_3},$$

$$c_3 = \sqrt{1+4a^2} - 2a, \quad c_4 = 1 - 2a + 2\sqrt{a(a-1)}, \quad c_5 = \frac{1}{c_4}.$$

We reduce the problem of determining the space—time changes in the thermal state of a semi-infinite body to the solution of the relaxation heat transfer equation (1) and the hyperbolic heat conduction equation (2), which we write in deviations of the parameters from the stationary value and in dimensionless space and time coordinates

$$\Delta q + \frac{\partial \Delta q}{\partial \mathrm{Fo}_r} = -K\frac{\partial \Delta\Theta}{\partial Y}, \quad \frac{\partial \Delta\Theta}{\partial \mathrm{Fo}_r} + \frac{\partial^2 \Delta\Theta}{\partial \mathrm{Fo}_r^2} = \frac{\partial^2 \Delta\Theta}{\partial Y^2} \tag{32}$$

for zero initial and generalized boundary conditions of the first-third kind [7,13]:

For the first kind

$$\Delta\Theta(0, \mathrm{Fo}_r) = \Delta\Theta_1; \tag{33}$$

For the second kind

$$-K\int_0^{\mathrm{Fo}_r} \frac{\partial \Delta\Theta(0, \mathrm{Fo}_r)}{\partial Y}\exp(\psi - \mathrm{Fo}_r)\,d\psi = \Delta q_1; \tag{34}$$

For the third kind

$$\frac{1}{\mathrm{Bi}_r}\int_0^{\mathrm{Fo}_r} \frac{\partial \Delta\Theta(0, \mathrm{Fo}_r)}{\partial Y}\exp(\psi - \mathrm{Fo}_r)\,d\psi = \Delta\Theta(0, \mathrm{Fo}_r) - \Delta t. \tag{35}$$

Boundary Conditions of the First Kind

Application of Laplace transform operations to (32) in the variable Fo_r by using the additional boundary conditions ($\partial\Delta\Theta(Y, 0)/\partial Fo_r = 0$, $\Delta\Theta(\infty, Fo_r) = 0$) yields the solution in the transform domain [14]:

$$\frac{\Delta\Theta(Y, s)}{\Delta\Theta_1} = \frac{1}{s}\exp\left[-Y\sqrt{s(s+1)}\right],$$

$$\frac{\Delta q(Y, s)}{K\Delta\Theta_1} = \frac{\exp\left[-Y\sqrt{s(s+1)}\right]}{\sqrt{s(s+1)}}. \tag{36}$$

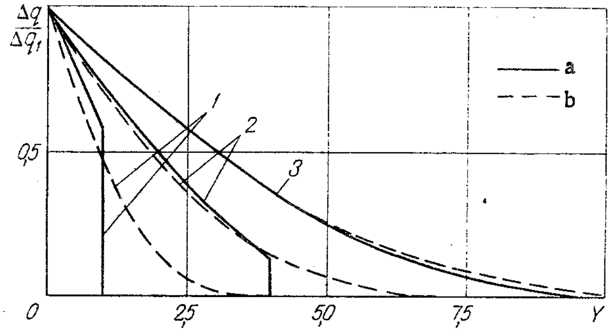

Fig. 2. Change in heat flux in a body for a jump in heat supply to its surface (see Fig. 1 for a and b): 1) Fo_r = 1; 2) 4; 3) 10.

The relations (15) and (18) in Table 2 permit writing the change in temperature and heat flux in the space–time domain

$$\frac{\Delta\Theta}{\Delta\Theta_1} = 2V_1 - V_{1,0}, \quad \frac{\Delta q}{K\Delta\Theta_1} = V_{1,0}, \tag{37}$$

where

$$V = V(\xi, \eta), \quad \xi = \frac{Fo_r + Y}{4}, \quad \eta = \frac{Fo_r - Y}{4}. \tag{38}$$

These dependences satisfy the boundary conditions and for $w_q \to \infty$ go over into the solution of the parabolic heat conduction equation [15].

The general pattern for the change in temperature in a semi-infinite body is shown in Fig. 1. Here results obtained on the basis of the classical relationship between the heat flux and temperature gradient (Fourier's law) are presented for comparison. It is seen from the figure that the boundedness of the heat propagation velocity in media governs the wave nature of the heat transport process; at any time an unperturbed domain and a domain of thermal changes exist. The front of the wave being propagated is determined by the expression $Y = Fo_r (y = w_q \tau)$. Hence, the temperature and heat flux undergo jumps on the front, which diminish exponentially with time

$$\frac{\Delta\Theta}{\Delta\Theta_1} = \frac{\Delta q}{K\Delta\Theta_1} = \exp(-0.5\,Fo_r) \text{ for } \quad Y = Fo_r. \tag{39}$$

It follows from (39) that the maximum heat flux occurs at the initial instant, whose magnitude is finite and determined by the level of the temperature jump on the body surface and by the coefficient K. This eliminates the fundamental contradiction for heat transport in the Fourier model. Analysis showed that the asymptotic representations of the solution (37) for large values of $\tau(Fo_r)$ go over into the classical dependences [15]. This is illustrated in Fig. 1: results obtained by the wave model of heat conduction and by the Fourier model approach each other as the dimensionless time increases. It is established by computations that both models yield slightly different results even for $Y \geq 10$ and $Fo_r \geq 10$.

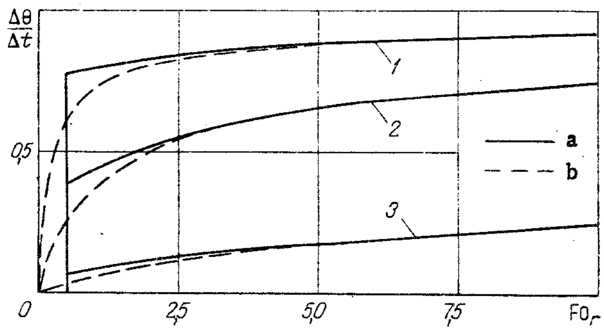

Fig. 3. Influence of external heat transfer conditions on the change in body temperature (Fig. 1 for a and b): 1) $Bi_r = \infty$; 2) 1; 3) 0.1.

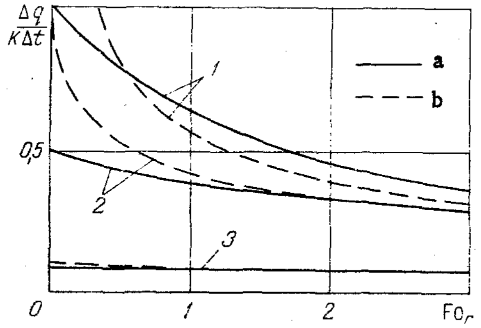

Fig. 4. Change in heat flux on a body surface for a jump in temperature of the environment (notation the same as in Fig. 3).

Boundary Conditions of the Second Kind

By using (34), the solution can be obtained in the transform domain [16]

$$\frac{\Delta\Theta(Y,\ s)}{\frac{1}{K}\Delta q_1} = \frac{1}{s}\sqrt{\frac{s+1}{s}}\exp[-Y\sqrt{s(s+1)}],$$

$$\frac{\Delta q(Y,\ s)}{\Delta q_1} = \frac{1}{s}\exp[-Y\sqrt{s(s+1)}]. \tag{40}$$

The time dependences of the change in temperature and heat flux are obtained by using the correspondences (18) and (20):

$$\frac{\Delta\Theta}{\frac{1}{K}\Delta q_1} = 4V_2 + V_{1,0},\quad \frac{\Delta q}{\Delta q_1} = 2V_1 - V_{1,0}, \tag{41}$$

where the arguments of the V-functions are determined by (38). The wave nature of the change in the thermal state of the body (Fig. 2) is also observed in the case of perturbation of the external heat supply. In contrast to the Fourier model, which yields a zero value of the change in surface temperature at the time of superposition of heat flux on the perturbation, taking account of the inertial term in the heat transport equation (32) results in a jump change in the temperature of the surface layer for $\tau = +0$ by the quantity $\Delta\Theta = (1/K)\Delta q_1$. This result also refines the information about the dynamic process of heat conduction in the initial stage.

From (41) it is easy to establish the agreement between the solutions using the wave heat conduction model and the Fourier model as $w_q \to \infty$ and $\tau \to \infty$.

Boundary Conditions of the Third Kind

The Laplace integral transform taking into account the conditions on the body surface (35) results in the following result in the transform domain:

$$\frac{\Delta\Theta(Y,\ s)}{\Delta t} = \frac{\mathrm{Bi}_r\sqrt{s(s+1)}\exp[-Y\sqrt{s(s+1)}]}{s[s+\mathrm{Bi}_r\sqrt{s(s+1)}]},$$

$$\frac{\Delta q(Y,\ s)}{K\Delta t} = \frac{\mathrm{Bi}_r\exp[-Y\sqrt{s(s+1)}]}{s+\mathrm{Bi}_r\sqrt{s(s+1)}}, \tag{42}$$

which by using (28) and (29) can be represented by the time dependences

$$\frac{\Delta\Theta}{\Delta t} = 2V_1 + \frac{\mathrm{Bi}_r}{1-\mathrm{Bi}_r}V_{1,0} - \frac{2}{1-\mathrm{Bi}_r^2}V_{1,1}, \tag{43}$$

$$\frac{\Delta q}{K\Delta t} = \frac{2\mathrm{Bi}_r}{1-\mathrm{Bi}_r^2}V_{1,1} - \frac{\mathrm{Bi}_r}{1-\mathrm{Bi}_r}V_{1,0}, \tag{44}$$

where $V_{1,1} = V_{1,1}(\xi,\ \eta,\ c_1)$; ξ, η are determined by (38), and $c_1 = (1 - \mathrm{Bi}_r)/(1 + \mathrm{Bi}_r)$.

The influence of the heat transfer conditions on the thermal change within the body and its surface is shown in Figs. 3 and 4, from which it is seen that the presence of an external thermal resistivity Bi_r diminishes the thermal action on the body. The amplitude of the temperature and thermal changes on the wave front decreases according to the law

$$\frac{\Delta\Theta}{\Delta t} = \frac{\Delta q}{K\Delta t} = \frac{\mathrm{Bi}_r}{1+\mathrm{Bi}_r}\exp(-0.5\mathrm{Fo}_r).$$

Therefore, the phenomenon of highly intensive heat transport can be described by the apparatus of special V-functions. The conditions $Y < 10$, $Fo_r < 10$ for which the wave and classical models of heat conduction yield differing results determine the space—time domain of the influence of relaxation effects on the heat conduction in media. Tables of integrals and correspondences of the transform-original, represented by V-functions, and a study of the fundamental properties of these functions permit analysis of the heat, moisture, mechanical and electrical energy transport phenomena under conditions of highly intensive changes in the transport substation fluxes. The presence of V-function tables [12] reduces engineering computations of the heat transport phenomenon to the simplest operations.

NOTATION

y, space coordinate, m; τ, time, sec; Θ, temperature, °K; q, heat flux density, W/m²; Δ, deviation of a parameter from the stationary value; a, thermal diffusivity, m²/sec; λ, heat conduction, W/(m·deg K); α, heat transfer coefficient, W/(m²·deg K); τ_{rq}, thermal relaxation time, sec; w_q, heat propagation velocity, m/sec; $Y = yw_q/a$, dimensionless coordinate; $Fo_r = \tau w_q^2/a$, dimensionless time (Fourier relaxation number); $Bi_r = \alpha a/\lambda w_q$, dimensionless heat transfer coefficient (Biot relaxation number); $K = \lambda w_q/a$; s, Laplace transform variable; I_n, n-th-order Bessel function of imaginary argument.

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